Math 352 Real Analysis Final

Instructions: Submit your work on any 5.

1. If $f: (0, 1) \to \mathbb{R}$ is uniformly continuous, show that $\lim_{x \to 0^+} f(x)$ exists. [20 pts]

2. Fix $y \in \ell_\infty$ and define $g: \ell_1 \to \ell_1$ by $g(x) = \{x_n y_n\}_{n=1}^{\infty}$. Show that g is uniformly continuous. [20 pts]

3. In any metric space, show that every open set is an F_σ and that every closed set is a G_δ. [20 pts]

4. Is there a dense open subset of $\mathbb{R}$ with uncountable complement? Explain. [20 pts]

5. Let $f_n: \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that $\{f_n\}$ converges uniformly on $\mathbb{Q}$. Show that $\{f_n\}$ actually converges uniformly on all of $\mathbb{R}$. [20 pts]

6. If $\sum_{n=1}^{\infty} |a_n| < \infty$, prove that $\sum_{n=1}^{\infty} a_n \sin nx$ and $\sum_{n=1}^{\infty} a_n \cos nx$ are uniformly convergent on $\mathbb{R}$. [20 pts]

7. Suppose that $f \in B[a, b]$. If $\bigvee_{a+\epsilon}^{b} f \leq M$ for all $\epsilon > 0$, does it follow that f is of bounded variation on $[a, b]$? Is $\bigvee_a^b f \leq M$? If not, what additional hypotheses on f would make this so? [20 pts]

8. If f has a continuous derivative on $[a, b]$, and P is any partition of $[a, b]$, show that $V(f, P) \leq \int_a^b |f'(t)| dt$. Hence, $\bigvee_a^b f \leq \int_a^b |f'(t)| dt$. [20 pts]

9. If f is continuous on $[1, n]$, compute $\int_1^n f(x) d[x]$, where $[x]$ is the greatest integer in x. What is $\int_1^t f(x) d[x]$ if t is not an integer? [20 pts]

10. If $\int_a^b f d\alpha = 0$ for every $f \in C[a, b]$, show that α is constant. [20 pts]

11. Show that $\cap \{\mathcal{R}_\alpha[a, b]: \alpha \ increasing\} = C[a, b]$ [20 pts]